2021 WMC 8 Solutions and Results

WMC Committee

This manual contains the answers, solutions, projected cutoffs, statistics, and results for the 2022 WMC 10, which ran from December 18 to January 15.

Answers: DDACD BADCE DBCAA CCBAB EEEEB

1. The Gettysburg Address began with the famous phrase "four score and seven years ago". How many years is that? (Note that there are 20 years in a score.)

   (A) 27 **(B)** 47 **(C)** 67 **(D)** 87 **(E)** 107

   Proposed by peace09

   Solution: Because there are 20 years in a score, there are $4 \cdot 20 = 80$ years in four score. Adding 7 more years yields a total of $\boxed{\textbf{(D)}}$ 87.

2. What is the sum of the digits of the quotient $3,431,969,149 \div 7$?

   (A) 7 **(B)** 16 **(C)** 25 **(D)** 34 **(E)** 43

   Proposed by peace09

   Solution: We may reinsert the commas in the dividend to our advantage, as follows: $343,196,91,49 \div 7$. Now, as $343 \div 7 = 49$, $196 \div 7 = 28$, $91 \div 7 = 13$, and $49 \div 7 = 7$, the desired quotient is $049,028,13,07$. The requested answer is therefore $4+9+2+8+1+3+7 = \boxed{\textbf{(D)}}$ 34.

3. Five siblings share a pie that is divided into 12 slices. The three youngest siblings eat 4 slices altogether, while the three oldest siblings eat 9 slices altogether. How many slices does the middle sibling eat?

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

   Proposed by peace09

   Solution: Let the five siblings eat a, b, c, d, and e slices, from the youngest sibling to the oldest. Accordingly, we may write

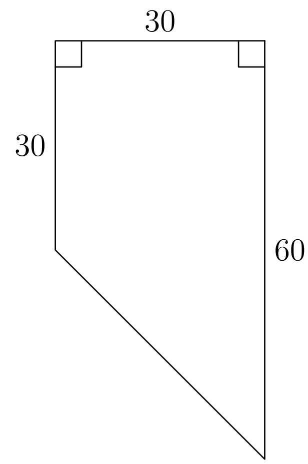
$$\begin{aligned} a+b+c+d+e &= 12 \\ a+b+c &= 4 \\ c+d+e &= 9. \end{aligned}$$

   Subtracting the first equation from the sum of the second and third, we have that

$$(a+b+c)+(c+d+e)-(a+b+c+d+e) = 4+9-12.$$

   Simplifying, $c = \boxed{\textbf{(A)}}$ 1, which is the requested answer.

4. Shown below is a student's sketch of the state of Nevada. Its western and northern borders are 30 millimeters long, and its eastern border is 60 millimeters long. What is the area of his sketch in square millimeters?



(A) 600 **(B)** 900 **(C)** 1350 **(D)** 1800 **(E)** 2700

Proposed by peace09

Solution: The sketch is a trapezoid with average of base lengths $\frac{30\text{ mm}+60\text{ mm}}{2} = 45$ mm and height 30 mm. Hence, its area is $45\text{ mm} \cdot 30\text{ mm} = \boxed{\textbf{(C)}}\ 1350\text{ mm}^2$.

5. Chirag sells 2 pencils and 3 pens for 5 dollars and 8 pens and 13 pencils for 21 dollars. For how many dollars does he sell 34 pencils and 55 pens?

   (A) 56 **(B)** 67 **(C)** 78 **(D)** 89 **(E)** 100

   Proposed by peace09

   Solution: Observe that if each pencil and pen costs \$1, the conditions are satisfied. And by the nature of independent linear systems of equations, we know that this is the only possibility, ergo $34 \cdot \$1 + 55 \cdot \$1 = \boxed{\textbf{(D)}}\ \$89$.

6. At Valley Vince's Burgers, a burger costs \$2.10. With Coupon A, Tyler can buy each burger for \$2.00, and with Coupon B, his grand total is decreased by \$1.80. What is the least number of burgers he must buy in order for Coupon A to be a better deal than Coupon B?

   (A) 18 **(B)** 19 **(C)** 20 **(D)** 21 **(E)** 22

   Proposed by peace09

   Solution: Compare the aggregate decrease in cost for each coupon. If Tyler buys n burgers, Coupon A decreases the cost by $n \cdot \$0.10$, and Coupon B decreases the cost by \$1.80. As a result, in order for Coupon A to be a better deal than Coupon B, we must have $n \cdot \$0.10 > \$1.80 \iff n \geq \boxed{\textbf{(B)}}\ 19$.

7. An artist mixes red and blue paint to produce 100 pints of what she calls "perplexing purple". If she adds 10 pints of red paint and 10 pints of blue paint, the mixture becomes 20 percent red paint. How many pints of blue paint are there in 150 pints of perplexing purple?

(A) 129 **(B)** 135 **(C)** 141 **(D)** 147 **(E)** 153

Proposed by peace09

Solution: Let there initially be r pints of red paint. The new mixture contains 120 total pints, of which $r+10$ are red. Then,

$$\frac{r+10}{120} = 20\% = \frac{1}{5} \implies r = \frac{120}{5} - 10 = 14.$$

Hence, perplexing purple is 14% red and 86% blue, and in 150 pints of perplexing purple, there are $86\% \cdot 150 = \boxed{\textbf{(A)}}$ 129 pints of blue paint.

8. Ashley divides a circular pizza into 4 slices so that each slice after the first is twice as large as the previous one. What is the degree measure of the central angle of the smallest slice?

(A) 15 **(B)** 18 **(C)** 20 **(D)** 24 **(E)** 30

Proposed by A1001

Solution: Let the four central angles have degree measures x, $2x$, $4x$, and $8x$. Accordingly, $x + 2x + 4x + 8x = 15x = 360$, wherefore $x = \boxed{\textbf{(D)}}$ 24.

9. Robin writes down the first N positive integers and removes all integers with the digit 0. If the remaining list is 111 integers long, what is N?

(A) 123 **(B)** 132 **(C)** 133 **(D)** 134 **(E)** 143

Proposed by peace09

Solution 1: From 1 to 100, we have 90 integers remaining—all but the multiples of 10. Henceforth, 101 to 110 are removed, as are 120 and 130, but 111 to 119 and 121 to 129 all remain. Hitherto we have $90 + 2 \cdot 9 = 108$ integers that remain. Therefore, 131 is the 109th integer, 132 the 110th, and $\boxed{\textbf{(C)}}$ 133 the 111th.

Solution 2: Observe that for each answer choice, the number of integers from $1, 2, \ldots, N$ with the digit 0 equals the number of integers with the digit 9. Hence, we can consider that Robin removes integers with the digit 9. Then, the remaining list can be seen as having integers base-9 counting upward from 1, so it is 111 integers long if the last integer is the 111th base-9 integer, $111_{\text{ten}} = \boxed{\textbf{(C)}}\ 133_{\text{nine}}$.[1]

[1] Due to **pog** (https://artofproblemsolving.com/community/c2829481h2740592p24443607).

10. Key, Fire, and Jupiter play a board game. Key plays first, then Fire, then Jupiter, then back to Key, and so on. At some point in the game, n turns have been played in total, and Fire has played 6 turns. What is the sum of all possible values of n?

(A) 18 **(B)** 35 **(C)** 36 **(D)** 37 **(E)** 54

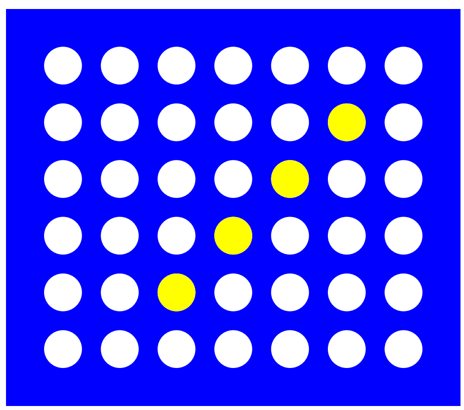
Proposed by peace09

Solution: We proceed with casework based on who last played his/her turn.

- If Key last played, the turn counts are 7, 6, and 6, so $n = 7 + 6 + 6 = 19$.
- If Fire last played, the turn counts are 6, 6, and 5, so $n = 6 + 6 + 5 = 17$.
- If Jupiter last played, the turn counts are 6, 6, and 6, so $n = 6 + 6 + 6 = 18$.

The requested answer is therefore $19 + 17 + 18 = \boxed{\textbf{(E)}}\ 54$.

11. A Connect Four grid has dimensions 6 by 7. A *four-in-a-row* is a set of 4 cells that are horizontally, vertically, or diagonally consecutive. How many four-in-a-rows are there in a Connect Four grid? Shown below is one possibility.



(A) 33 **(B)** 45 **(C)** 57 **(D)** 69 **(E)** 81

Proposed by peace09

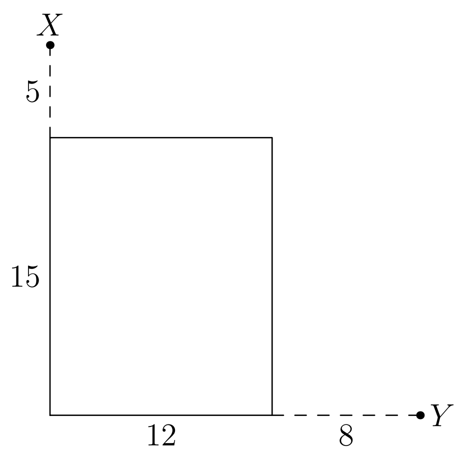
Solution: Observe that each four-in-a-row is uniquely determined by its left and/or bottom-most cell, as well as its type—eastward, northward, northeastward, or northwestward. Accordingly, we proceed with casework based on type.

- *Case 1:* eastward. Then, in order for the four-in-a-row to remain within the boundaries of the grid, the leftmost cell can and must lie within the leftmost 6 by 4 subgrid. Hence, we have $6 \cdot 4 = 24$ four-in-a-rows here.
- *Case 2:* northward. Similarly to Case 1, we see that the bottommost cell can and must lie within the bottommost 3 by 7 subgrid. Hence, we have $3 \cdot 7 = 21$ four-in-a-rows here.

- *Case 3:* northeastward. By nature of the diagonal four-in-a-row, here our possibilities are more limited. Indeed, the left and bottom-most cell must lie within the left and bottom-most 3 by 4 subgrid, so we have only $3 \cdot 4 = 12$.
- *Case 4:* northwestward. This is symmetric to Case 3, so 12 again.

Ergo $24 + 21 + 12 + 12 =$ **(D)** 69.

12. A building has dimensions 12 miles by 15 miles, and points X and Y are located 5 miles north and 8 miles east of the building respectively, as shown below. In a race from X to Y, Nick the ghost can drift through the building, while Harry the human must run around it, and both take optimal routes. To the nearest integer, how many miles shorter is Nick's route than Harry's?



(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Proposed by peace09

Solution: We consider each route separately.

- Nick can travel directly from X to Y, $12 + 8 = 20$ horizontal miles and $5 + 15 = 20$ vertical, for a total $20\sqrt{2}$ miles.
- Harry must first travel from X to the northeastern corner of the building, for $\sqrt{5^2 + 12^2} = 13$ miles, then from the northeastern corner to Y, for $\sqrt{15^2 + 8^2} = 17$ miles. Adding, we have a total $13 + 17 = 30$ miles for Harry.

Hence. Nick's route is $30 - 20\sqrt{2}$ miles shorter than Harry's. Using $\sqrt{2} \approx 1.4$, we can approximate this as $30 - 20 \cdot 1.4 =$ **(B)** 2.

13. Ms. White assigns the numbers 1, 2, 3, 4, 5, and 6 to Alice, Bob, Carol, David, Eve, and Frederick in some order, so that each student knows their own number, but none of the others' numbers. She proceeds to announce the following:

- Alice's number is smaller than Bob's number,
- Carol's number is the square of David's number, and
- Eve's number is larger than Frederick's number.

Frederick promptly exclaims, "I know what Eve's number is!" What is the sum of Alice's Carol's, and Eve's numbers?

(A) 6 **(B)** 9 **(C)** 11 **(D)** 13 **(E)** 15

Proposed by peace09

Solution: We consider each announcement one by one.

- In order for Frederick to know Eve's number e, there must be exactly one number in $\{1, 2, 3, 4, 5, 6\}$ greater than Frederick's f, so $f = 5$ and $e = 6$.
- Carol's c is the square of David's d, and since c and d are distinct elements of $\{1, 2, 3, 4, 5, 6\}$, we have $d = 2$ and $c = 4$.
- The only two remaining elements for b and a are 3 and 1, which we assign in precisely that order.

Ergo $(a, b, c, d, e, f) = (1, 3, 4, 2, 6, 5)$ and $a + c + e = \boxed{\textbf{(C)}}\ 11$.

14. In the sport basketball, players can score 2-point shots, 3-point shots, or free throws (each worth 1 point). In a basketball game, a team scores a 2-point shots, b 3-point shots, and 17 free throws for a final score of 112. How many possible values are there for a?

    (A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

    Proposed by peace09

    Solution: We may write the following equation:

    $$2a + 3b + 17 = 112 \implies 2a + 3b = 95.$$

    Consider b, which must be at least 0 and at most 31 (as $3 \cdot 32 > 95$). Because $2a$ is even, $3b$ and thus b must be odd, and there are $\boxed{\textbf{(A)}}$ 16 odd integers between 0 and 31 inclusive. Since each b corresponds to exactly one a, this is the answer.

15. Two increasing sequences of positive integers both have first term 1. In one sequence, the mean of the first k terms is k, and in the other sequence, the range of the first k terms is k, for every positive integer $k > 1$. What is the difference between the 100th terms of the sequences?

    (A) 98 **(B)** 99 **(C)** 100 **(D)** 101 **(E)** 102

Proposed by peace09

Solution: In the first sequence, in order for the mean of the first 2 terms to be 2, the second term must be 3, since the first term is 1. Next, in order for the mean of the first 3 terms to be 3, the third term must be 5, since the first two terms are 1 and 3. Continuing in this fashion yields

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19, \ldots,$$

which has 100th term 199.

In the second sequence, in order for the range of the first 2 terms to be 2, the second term must (again) be 3, since the first term is 1. But then the second term becomes 4, the third 5, and so on. So we have

$$1, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots,$$

which has 100th term 101.

The requested answer is therefore $199 - 101 = \boxed{\textbf{(A)}}\ 98$.

16. Richard's unfair coin lands heads with four times the probability that it lands tails twice in a row. He will flip the coin once to see whether it lands heads or tails, and Vanessa, assuming the coin is fair, bets it will land tails. The probability that she wins the bet can be expressed in the form $\frac{\sqrt{a}-b}{c}$, where a, b, and c are positive integers and a is square-free. What is $a + b + c$?

    (A) 20 **(B)** 23 **(C)** 26 **(D)** 29 **(E)** 32

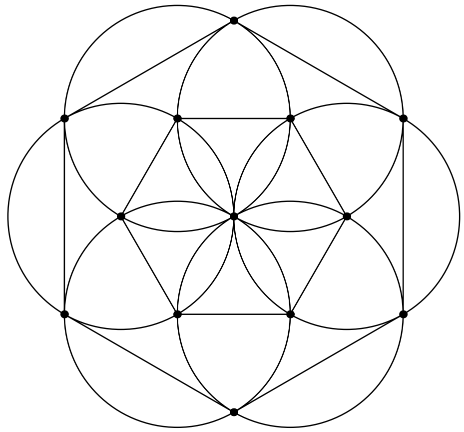
    Proposed by A1001

    Solution: If p is the probability she wins the bet, then $1-p$ is the probability it lands heads, and p^2 the probability that it lands tails twice in a row. Hence,

    $$1 - p = 4p^2 \implies 4p^2 + p - 1 = 0.$$

    By the quadratic formula, $p = \frac{\sqrt{17}-1}{8}$ (as $p > 0$), yielding $17 + 1 + 8 = \boxed{\textbf{(C)}}\ 26$.

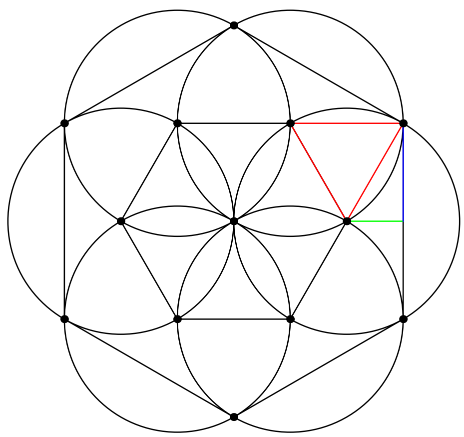
17. Six circles are centered at the vertices of a regular hexagon, as shown below. Thirteen intersection points result, one at the center and six at the vertices of the hexagon. The remaining six points determine another hexagon with an area equalling how many times the area of the original hexagon?



(A) 2 **(B)** $\frac{9}{4}$ **(C)** 3 **(D)** $\frac{16}{9}$ **(E)** 4

Proposed by peace09

Solution: Refer to the diagram below.



Without loss of generality, let the original hexagon be unit, so that the red segments each have length 1. The red-green-blue triangle is a $30-60-90$ triangle, so the green segment has length $\frac{1}{2}$, and the blue segment $\frac{\sqrt{3}}{2}$. In particular, it follows that the new hexagon has side length $\sqrt{3}$, and thus an area of $(\sqrt{3})^2 = \boxed{\textbf{(C)}}$ 3 times that of the original triangle.

18. The ten digits from 0 to 9 are permuted and combined to form an integer. Two such integers are 0981276345 and 9018723654. What is the probability that this integer is a multiple of 2^1, 3^2, and 5^3?

(A) $\frac{1}{720}$ **(B)** $\frac{1}{360}$ **(C)** $\frac{1}{240}$ **(D)** $\frac{1}{180}$ **(E)** $\frac{1}{144}$

Proposed by peace09

Solution: First, in order for the integer to be a multiple of $2^1 = 2$, it is necessary and sufficient for the last digit to be even.

Next, in order for the integer to be a multiple of $3^2 = 9$, the sum S of the digits has to be a multiple of 9. But $S = 0 + 1 + \cdots + 9 = 45$, a multiple of 9, so this condition is always satisfied.

Then, in order for the integer to be a multiple of $5^3 = 125$, the last three digits must be 000, 125, 250, 375, 500, 625, 750, or 875. As all the digits are distinct and the last digit is even, the only possibilities are 250 and 750.

All in all, the given condition is satisfied if and only if the last three digits are 250 or 750. Hence, out of 10! total outcomes, there are $2 \cdot 7!$ desired outcomes: 2 ways to choose the last three digits, and 7! to order the first seven. The requested probability is therefore $\frac{2\cdot7!}{10!} = \frac{2}{8\cdot9\cdot10} = \boxed{\textbf{(B)}}\ \frac{1}{360}$.

19. Sarah writes down the numbers 2 and 3 on a whiteboard. Every minute, she writes down the product of the previous two numbers. Hence, the next two numbers she writes down are $2 \times 3 = 6$ and $3 \times 6 = 18$, and the fifth number is $6 \times 18 = 108$. How many divisors does the tenth number have?

(A) 770 **(B)** 910 **(C)** 1470 **(D)** 1610 **(E)** 1960

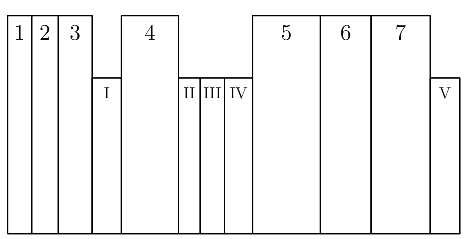
Proposed by peace09

Solution: Write all numbers on the whiteboard in terms of their prime factorization, so the first five numbers are

$$2 = 2^1 \cdot 3^0, 3 = 2^0 \cdot 3^1, 6 = 2^1 \cdot 3^1, 18 = 2^1 \cdot 3^2, \text{ and } 108 = 2^2 \cdot 3^3.$$

Continuing in this fashion yields $2^3 \cdot 3^5$, $2^5 \cdot 3^8$, and $2^8 \cdot 3^{13}$, whose exponents we recognize as the Fibonacci numbers. Further we have $2^{13} \cdot 2^{21}$ and $2^{21} \cdot 3^{34}$ as the ninth and tenth numbers; the latter has $(21 + 1)(34 + 1) = \boxed{\textbf{(A)}}$ 770 divisors.

20. In how many ways can the 7 Harry Potter books and the 5 Percy Jackson books be displayed on a bookshelf so that both the Harry Potter books and the Percy Jackson books are in order? One such ordering is shown below.



(A) 495 **(B)** 792 **(C)** 924 **(D)** 95040 **(E)** 3991680

Proposed by peace09

Solution: We can choose the slots for the Percy Jackson books in $\binom{12}{5} = 792$ ways, thence their order is fixed (ordered in increasing volume number). But then the slots for the Harry Potter books are also fixed, being the remaining 7 slots, and their order too is fixed in a similar fashion. So the answer is simply $\boxed{\textbf{(B)}}$ 792.

21. A circle with integer area intersects a rectangle with side lengths 2 and 6 at 8 points. What is the sum of the smallest and largest possible areas of the circle?

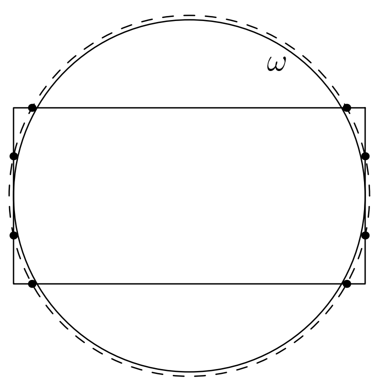
(A) 56 **(B)** 57 **(C)** 58 **(D)** 59 **(E)** 60

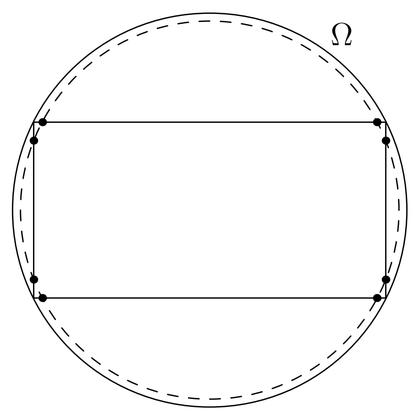
Proposed by peace09

Solution: The smallest such circle is that slightly larger than the circle ω tangent to the sides of length 2, as shown at left below. As ω has area $(\frac{6}{2})^2\pi = 9\pi$, which is approximately 28.26, the smallest possible area is 29.

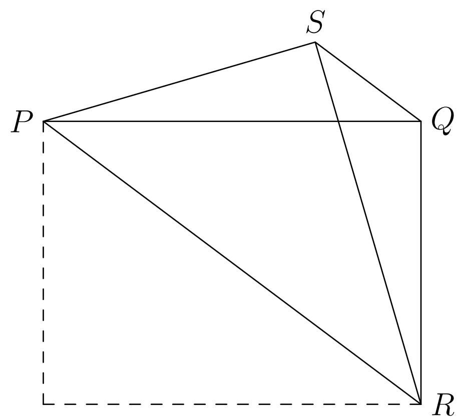
Meanwhile, the largest such circle is that slightly smaller than the circle Ω circumscribing the rectangle, as shown at right below. As Ω has area $(\frac{\sqrt{2^2+6^2}}{2})^2\pi = 10\pi$, which is approximately 31.41, the largest possible area is 31.

The requested answer is therefore $29 + 31 = \boxed{\textbf{(E)}}$ 60.





22. Rohan labels the vertices of a 3 cm by 4 cm sheet of paper as P, Q, R, and S, in that order going clockwise, so that $\overline{PQ}$ is along the long side. Then, he folds the rectangle across $\overline{PR}$ as shown below to determine whether or not it is a line of symmetry. He finds it isn't, and proceeds to measure the length QS to see by how much the fold was "inaccurate". What length in centimeters should he obtain?



(A) 0.6 **(B)** 0.8 **(C)** 1 **(D)** 1.2 **(E)** 1.4

Proposed by peace09

Solution 1: Let X be the intersection of $\overline{PQ}$ and $\overline{RS}$. Observe that $\triangle QXS$ and $\triangle PXR$ are similar; we shall compute the ratio of similarity between them.

Let $QX = SX = x$, so that $PX = RX = 4 - x$ (as $PQ = RS = 4$). Then by Pythagoras on $\triangle QXR$,

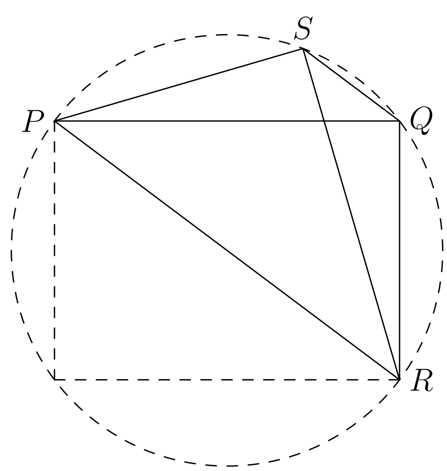
$$x^2 + 3^2 = (4 - x)^2 = x^2 - 8x + 16 \implies x = \frac{16 - 3^2}{8} = \frac{7}{8}.$$

As a result, $PX = 4 - \frac{7}{8} = \frac{25}{8}$, so the aforementioned similarity ratio is $\frac{QX}{PX} = \frac{7/8}{25/8} = \frac{7}{25}$. Hence, $\frac{QS}{PR} = \frac{QS}{5} = \frac{7}{25} \implies QS = \frac{7}{5} = \boxed{\textbf{(E)}}\ 1.4$.

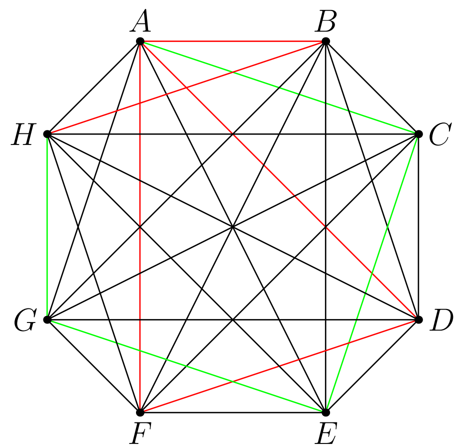
Solution 2: As $\angle PQR = \angle PSR = 90°$, points Q and S lie on the circle with diameter $\overline{PR}$; in particular, $PRQS$ is cyclic. Therefore, by Ptolemy's theorem,

$$PR \cdot QS + PS \cdot QR = PQ \cdot RS \implies 5QS + 3 \cdot 3 = 4 \cdot 4.$$

Solving gives us $QS = \frac{4^2 - 3^2}{5} = \frac{7}{5}$, answer choice $\boxed{\textbf{(E)}}$.



23. The sides and diagonals of convex octagon $ABCDEFGH$ are drawn. How many paths are there starting at A and ending at H from vertex to vertex along the drawn segments so that no vertex is visited more than once? The diagram below shows a valid path in green and an invalid path in red.



(A) 1797 **(B)** 1832 **(C)** 1877 **(D)** 1912 **(E)** 1957

Proposed by peace09

Solution: We proceed with casework on the length of the path. For example, for paths containing 8 vertices, there are $\binom{6}{6}$ ways to choose the intermediate points (from B to G), and $6!$ ways to order them. Similarly, for paths of length 7, there are $\binom{6}{5}$ ways to choose the intermediate points and $5!$ ways to order them. Continuing in this fashion yields

$$\binom{6}{6}\cdot 6! + \binom{6}{5}\cdot 5! + \binom{6}{4}\cdot 4! + \binom{6}{3}\cdot 3! + \binom{6}{2}\cdot 2! + \binom{6}{1}\cdot 1! + \binom{6}{0}\cdot 0!$$
$$= 1\cdot 720 + 6\cdot 120 + 15\cdot 24 + 20\cdot 6 + 15\cdot 2 + 6\cdot 1 + 1\cdot 1$$
$$= 720 + 720 + 360 + 120 + 30 + 6 + 1,$$

which evaluates to $\boxed{\textbf{(E)}}$ 1957.

24. A positive integer N is called *plentiful* if the product of its proper divisors (all divisors other than itself) is greater than N. For example, the number 12 is plentiful, because its proper divisors 1, 2, 3, 4, and 6 multiply to 144, which is greater than 12. How many of the first 50 positive integers are plentiful?

(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

Proposed by peace09

Solution: We compute a closed form for the product of the proper divisors of N. Let $a_1, a_2, \ldots, a_k$ be the divisors of N, with $a_1 = 1$ and $a_k = N$. In the product $a_1 a_2 \ldots a_k$, we can "pair up" the factors $a_1 a_k$, $a_2 a_{k-1}$, and so on, each of which multiply to N. There are $\frac{k}{2}$ such pairs,[2] so the whole product evaluates to $N^{k/2}$, and dividing by $a_k = N$ (the only divisor that isn't proper) yields $N^{k/2-1}$.

Hence, N is plentiful if and only if $N^{k/2-1} > N$; that is, $\frac{k}{2} - 1 > 1$ or $k > 4$.

- If $k = 5$, N must be of the form p^4. The only possibility is $N = 2^4 = 16$.
- If $k = 6$, N can be of the form p^5 or p^2q. The former has 1 possibility ($2^5 = 32$), and the latter 7 ($2^2 \cdot 3 = 12$, $2^2 \cdot 5 = 20$, $2^2 \cdot 7 = 28$, $2^2 \cdot 11 = 44$, $3^2 \cdot 2 = 18$, $3^2 \cdot 5 = 45$, and $5^2 \cdot 2 = 50$).
- If $k = 7$, N must be of the form p^6. As $2^6 > 50$, there are no possibilities.
- If $k = 8$, N can be of the form p^7, p^3q, or pqr. The first has no possibilities, the second has 2 ($2^3 \cdot 3 = 24$ and $2^3 \cdot 5 = 40$), and the third 2 ($2 \cdot 3 \cdot 5 = 30$ and $2 \cdot 3 \cdot 7 = 42$).
- If $k = 9$, N can be of the form p^8 or p^2q^2. The former has no possibilities and the latter 1 ($2^2 \cdot 3^2 = 36$).
- If $k = 10$, N can be of the form p^9 or p^4q. The former has no possibilities and the latter 1 ($2^4 \cdot 3 = 48$).

Clearly higher values of k will yield values for N too large, and therefore the requested answer is $1 + 8 + 0 + 4 + 1 + 1 =$ **(E)** 15.

25. For any positive integer n, let $C(n)$ be the number of digits in n, and let $S(n)$ be the sum of its digits. How many positive integers n satisfy $C(n)^{C(n)} = S(n)$?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Proposed by peace09

Solution: We proceed with casework on $C(n)$.

- If $C(n) = 1$ so $S(n) = 1^1 = 1$, the only possibility is clearly $n = 1$.
- If $C(n) = 2$ so $S(n) = 2^2 = 4$, we have 4 possibilities in $n = 13, 22, 31, 40$.
- If $C(n) = 3$ so $S(n) = 3^3 = 27$, we have 1 possibility in $n = 999$.

Then by experimentation, it appears there are no solutions n with $C(n) > 3$. To see why, observe that given a fixed $C(n)$, the largest possible digit sum $S(n)$ is $9C(n)$, when n consists of all 9's ($n = 99\ldots9$). Hence $S(n) \le 9C(n)$, and plugging in $S(n) = C(n)^{C(n)}$ yields $C(n)^{C(n)} \le 9C(n)$, or $C(n)^{C(n)-1} \le 9$. Now it is evident that we must have $C(n) \le 3$, and so the requested answer is **(B)** 6.

[2] If k is odd, so $\frac{k}{2}$ pairs is nonsensical, there are instead $\frac{k-1}{2}$ pairs multiplying to N, and a lone factor of $\sqrt{N} = N^{1/2}$ (where N is thus a perfect square). Multiplying gives $N^{(k-1)/2} \cdot N^{1/2} = N^{k/2}$.

Contest Summary: Thank you to all 28 people who submitted, for making our first year's contest a success! Congratulations to the following top scorers:

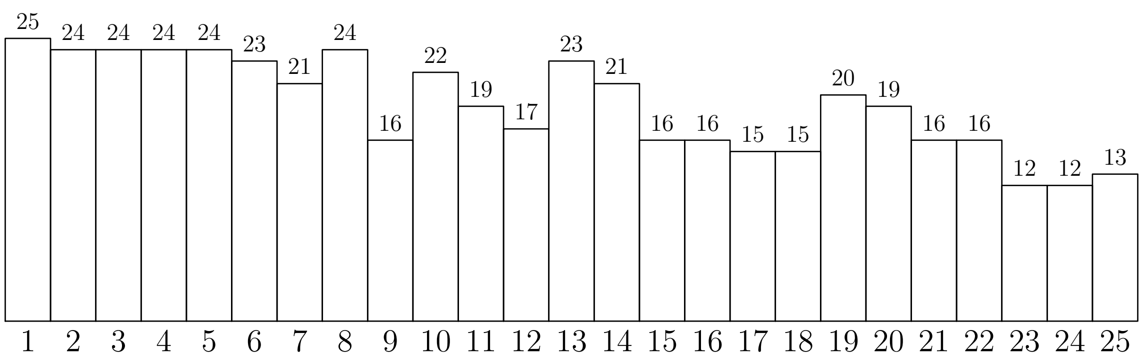
1. RedFireTruck, stayhomedomath, DeToasty3 (25/25)

2. vsamc, MathWhiz214 (24/25)

3. lrjr24, exp-ipi-1, capybara42 (23/25)

Comprehensive results can be accessed on the page that follows.

Solves per Problem:



Cutoffs:[3]

- Average Score: 8
- Honor Roll: 14
- Distinguished Honor Roll: 20

[3]Projected and based on personal opinion and feedback. Consider them with a grain of salt.

Username	Score
RedFireTruck	25
stayhomedomath	25
DeToasty3	25
vsamc	24
MathWhiz214	24
lrjr24	23
exp-ipi-1	23
capybara42	23
kante314	22
bobthegod78	22
pi271828	22
dragnin	22
wxl18	22
Anonymous	21
Anonymous	20
Anonymous	17
Anonymous	16
smartguy888	16
Anonymous	15
WhitePhoenix	15
aops-g5-gethsemanea2	14
Anonymous	13
ijco	13
Anonymous	12
Anonymous	11
samrocksnature	5
centslordm	2
math31415926535	1
pog	0